(This is a .pdf version of the tutorial posted here)

To explain this filter as a purely real filter (not an implementation of a complex filter) would require a longer explanation of the z-transform, as the simple approach I gave above doesn't really work in this case. The reason being is that the filter needs to be described using two difference equations (because of that one-sample feedback loop in the imaginary section that is within the overall two-sample loop). For the sake of clarity, I'm going to make these substitutions/names:

$c_1 = r\cos\omega$

$c_2 = r\sin\omega$

y_1 = real output

y_2 = imaginary output

The equations become:

$$y_1[n] = x[n] + c_1 y_1[n-1] - c_2 y_2[n-1]$$
$$y_2[n] = c_2 y_1[n-1] + c_1 y_2[n-1]$$

We can see one sample delays here, but they're in two equations. If we find the transfer function that gives us $y_1[n]$, we can see exactly the order of that specific section, and even find a way to implement it without the need for $y_2[n]$. However, the z-transform as explained earlier is just a shortcut for when you have a simple single-difference-equation filter. It doesn't cut it here. You have to do it the long way.

THE Z-TRANSFORM-extended

When we are in the z-domain, $X(z)$ is the z-transform of the input, and $Y(z)$ is the z-transform of the output (z itself can be an arbitrary complex number that you can plug in to find useful information about the system, but we don't care here. It's just a variable). Since we are assuming an arbitrary input, we don't know what either of these are, nor to we really care about that either. What we are trying to find is what causes $X(z)$ to become $Y(z)$. That is $H(z)$: the transfer function of the filter. In the z-domain, the basic formula looks like this:

$$Y(z) = H(z)X(z)$$

In other words, the z-transform of the input times the z-transform of the filter function gives us the z-transform of the output. So, of course, a little algebra gives the basic formula of the transfer function:

$$H(z) = \frac{Y(z)}{X(z)}$$

Remember that. This is the goal when implementing the z-transform.

In the time domain, some thing like $x[n-k]$ is a mathematical way of saying "the input delayed by k samples". In the z-domain, the same thing is said as $z^{-k}X(z)$. z^{-k} applies a delay of k samples, and $X(z)$ is the z-transformed signal that the delay is applied to. This is really all the z-transform itself is. Once we do that, it's just simple algebra to get to $H(z)$.

To illustrate, let's do a simple example:

$$y[n] = b_0 x[n] + b_1 x[n-1] + a_1 y[n-1]$$

First, we do the transformation:

$$Y(z) = b_0 X(z) + b_1 z^{-1} X(z) + a_1 z^{-1} Y(z)$$

Again, we're trying to find $H(z) = Y(z)/X(z)$ here, so we need to rearrange it to get $Y(z)/X(z)$ on one side.

$$Y(z) - a_1 z^{-1} Y(z) = b_0 X(z) + b_1 z^{-1} X(z)$$
$$Y(z)\left(1 - a_1 z^{-1}\right) = X(z)\left(b_0 + b_1 z^{-1}\right)$$
$$\frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1}}{1 - a_0 z^{-1}} = H(z)$$

And there you have it. As you can see, that lines up with the simple explanation given earlier, and also explains why the $y[n]$ coefficients change sign while the $x[n]$ ones don't.

Hopefully you can also see why it is so useful for manipulating compared to difference equations. In the time domain, the delay and the signal are linked. $y[n]$ and $y[n-1]$ are separate things. However, in the z-domain, $Y(z)$ and $z^{-1}Y(z)$ allow you to extract $Y(z)$. You can pull the signal away from the delay!

APPLYING TO MULTIPLE DIFFERENCE EQUATIONS

So let's see how to use our newfound knowledge of the z-transform to find the transfer function of a filter that requires multiple difference equations using the one katjav provided. I'll copy the equations again here so you don't have to scroll up:

$$y_1[n] = x[n] + c_1 y_1[n-1] - c_2 y_2[n-1]$$
$$y_2[n] = c_2 y_1[n] + c_1 y_2[n-1]$$

We want to know the transfer function that gives us the output $y_1[n]$. First things first, let's transform and roll out! (You knew a Transformers reference was coming at some point, let's be real with ourselves.)

$$Y_1(z) = X(z) + c_1 z^{-1} Y_1(z) - c_2 z^{-1} Y_2(z)$$
$$Y_2(z) = c_2 Y_1(z) + c_1 z^{-1} Y_2(z)$$

We want to find the $H(z)$ for the output that gives us $y_1[n]$, so we need to get to $Y_1(z)/X(z)$. But first, we need to isolate $Y_2(z)$ in the second equation:

$$Y_2(z) = c_2 Y_1(z) + c_1 z^{-1} Y_2(z)$$
$$Y_2(z) - c_1 z^{-1} Y_2(z) = c_2 Y_1(z)$$
$$Y_2(z)\left(1 - c_1 z^{-1}\right) = c_2 Y_1(z)$$
$$Y_2(z) = \left(\frac{c_2}{1 - c_1 z^{-1}}\right) Y_1(z)$$

Now that we know $Y_2(z)$, we can substitute it into the first equation and find $Y_1(z)/X(z)$. It gets a little messy here, so I'll try not to skip any not-so-obvious steps, for clarity. Hopefully you can follow. Let me know if I screw up somewhere or if something isn't clear:

$$Y_1(z) = X(z) + c_1 z^{-1} Y_1(z) - \frac{(c_2 z^{-1})c_2}{1 - c_1 z^{-1}} Y_1(z)$$

$$Y_1(z) - c_1 z^{-1} Y_1(z) + \frac{c_2{}^2 z^{-1}}{1 - c_1 z^{-1}} Y_1(z) = X(z)$$

$$Y_1(z)\left(1 - c_1 z^{-1} + \frac{c_2{}^2 z^{-1}}{1 - c_1 z^{-1}}\right) = X(z)$$

$$\frac{Y_1(z)}{X(z)} = \frac{1}{\left(1 - c_1 z^{-1} + \frac{c_2{}^2 z^{-1}}{1 - c_1 z^{-1}}\right)}$$

$$\frac{Y_1(z)}{X(z)} = \frac{1}{\left(\frac{1 - c_1 z^{-1}}{1 - c_1 z^{-1}} - \frac{\left(1 - c_1 z^{-1}\right) c_1 z^{-1}}{1 - c_1 z^{-1}} + \frac{c_2{}^2 z^{-1}}{1 - c_1 z^{-1}}\right)}$$

$$\frac{Y_1(z)}{X(z)} = \frac{1 - c_1 z^{-1}}{1 - c_1 z^{-1} - c_1 z^{-1} + c_1{}^2 z^{-2} + c_2{}^2 z^{-2} + c_2{}^2 z^{-1}}$$

$$\frac{Y_1(z)}{X(z)} = \frac{1 - c_1 z^{-1}}{1 + \left(c_2{}^2 - 2c_1\right) z^{-1} + c_2{}^2 z^{-2}}$$

And there you have it. Assuming I did that right, that's a second-order filter with two poles and one zero. At least, it is in the z-domain. That's why I said "I *believe* the short answer is yes" in the beginning and then made you read through all of that. ;-p

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